LECTURE 2: CHEBYSHEV APPROXIMATION THEORY AND STRUCTURE

2.1. Preliminary remarks, Chebyshev zeros and Integration

"Chebyshev polynomials are everywhere dense in Numerical analysis"

This remark has been attributed to a number of distinguished mathematicians and numerical analysts. There is almost not any area of numerical analysis where Chebyshev polynomials do not drop in like surprise visitors. Indeed, there are now a number of subjects in which these polynomials take a significant position in modern developments including orthogonal polynomials, polynomial approximation, numerical integration and spectral methods for PDEs.

There are several kinds of Chebyshev polynomials. In particular we will introduce the first kind $T_n(x)$ and second kind $U_n(x)$ as well as a pair of related (Jacobi) polynomials $V_n(x)$ and $W_n(x)$, which we call the Chebyshev polynomials of the third and fourth kinds respectively. In addition we have the shifted polynomials $T_n^*(x), U_n^*(x), V_n^*(x), W_n^*(x)$.

Clearly some definition of Chebyshev polynomials is needed right away and therefore we use as our primary definitions which is related to trigonometric functions.

Definition 1.1: The Chebyshev polynomial $T_n(x)$ of the first kind and second kind $U_n(x)$ are the polynomials in x of degree *n* defined by the relations

$$T_n(x) = \cos n\theta,\ x = \cos\theta, \tag{2.1}$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},\ x = \cos\theta. \tag{2.2}$$

If the range of the variable x is the interval then the range of the corresponding variable θ can be taken as $[0,\pi]$. These ranges are traversed in opposite direction, since $x=-1$ corresponds to $\theta=\pi$ and $x=1$ corresponds to $\theta=0$.

In practice, it is neither convenient nor efficient to work out each $T_n(x)$ from first principles. Rather by combining the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = \cos\theta\cos(n-1)\theta,$$

with Definition 1.1, we obtain the fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-1}(x),\quad T_0(x)=1,\ T_1(x)=x,\ n=2,3,\ldots. \tag{2.3}$$

Similarly, by combining the trigonometric identity

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\cos\theta\sin n\theta,$$

leads us to a relationship

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),\quad U_0(x)=1,\ U_1(x)=2x,\ n=2,3,\ldots. \tag{2.4}$$

We may immediately deduce from (2.1)-(2.4) that first few Chebyshev polynomials of the first and second kind are:

Table 1: Chebyshev polynomials of the first and second kind

n	$T_n(x)$	$U_n(x)$
0	1	1
1	x	$2x$
2	$2x^2-1$	$4x^2-1$
3	$4x^3-3x$	$8x^3-4x$
4	$8x^4-8x^2+1$	$16x^4-12x^2+1$
5	$16x^5-20x^3+5x$	$32x^{5k}-32x^3+6x$
6	$32x^6-48x^4+18x^2-1$	$64x^6-80x^4+24x^2-1$

It is easy to deduce from (2.3)-(2.4) that the leading coefficient of x^n in $T_n(x)$ and $U_n(x)$ are 2^{n-1} and 2^n respectively.

Definition 1.2: The Chebyshev polynomial of the third kind $V_n(x)$ and fourth kind $W_n(x)$ are the polynomials in x of degree n defined respectively by

$$V_n(x)=\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{\theta}{2}\right)},\ x=\cos\theta, \tag{2.5}$$

$$W_n(x)=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)},\ x=\cos\theta. \tag{2.6}$$

These polynomials are sometimes referred to as the "airfoil polynomials" but Gautschi (1992) named them the "third" and "fourth" kind Chebyshev polynomials.

Since

$$\cos\left(n+\frac{1}{2}\right)\theta+\cos\left(n-2+\frac{1}{2}\right)\theta=2\cos\theta\cos\left(n-1+\frac{1}{2}\right)\theta,$$
$$\sin\left(n+\frac{1}{2}\right)\theta+\sin\left(n-2+\frac{1}{2}\right)\theta=2\cos\theta\sin\left(n-1+\frac{1}{2}\right)\theta,$$

implies that

$$\begin{aligned}&V_n(x)=2xV_{n-1}(x)-V_{n-2}(x),\quad V_0(x)=1,\ V_1(x)=2x-1,\ n=2,3,\ldots\\&W_n(x)=2xW_{n-1}(x)-W_{n-2}(x),\quad W_0(x)=1,\ W_1(x)=2x+1,\ n=2,3,\ldots\end{aligned}. \tag{2.7}$$

From the Definition 1.2 and Eq. (2.7) we can list down few terms of Chebyshev polynomials of the third and fourth kinds

Table 2: Chebyshev polynomials of the third and forth kind

n	$V_n(x)$	$W_n(x)$
0	1	1
1	$2x-1$	$2x+1$
2	$4x^2-2x-1$	$4x^2+2x-1$
3	$8x^3-4x^2-4x+1$	$8x^3+4x^2-4x-1$
4	$16x^4-8x^3-12x^2+4x+1$	$16x^4+8x^3-12x^2-4x+1$
5	$32x^5-16x^4+32x^3+12x^2+6x-1$	$32x^5-16x^4-32x^3-12x^2+6x+1$
6	$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1$	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$

Note that V_n and W_n are neither even nor odd (unlike T_n and U_n). We have seen that the leading coefficient of x^n is 2^n in both V_n and W_n.

Properties and relationships

We can deduce relationship of Chebyshev polynomials

$$\begin{aligned}
&V_n(x)=U_n(x)-U_{n-1}(x),\\
&W_n(x)=U_n(x)+U_{n-1}(x),\\
&U_n(x)=\frac{1}{2}\left[V_n(x)+W_n(x)\right],.\\
&V_n(x)+V_{n-1}(x)=2T_n(x),\\
&W_n(x)-W_{n-1}(x)=2T_n(x).
\end{aligned} \tag{2.8}$$

Evaluation of the product

$$\begin{aligned}
&T_m(x)T_n(x)=\frac{1}{2}\left(T_{m+n}(x)+T_{|m-n|}(x)\right),\\
&xT_n(x)=\frac{1}{2}\left(T_{n+1}(x)+T_{|n-1|}(x)\right),\\
&xU_n(x)=\frac{1}{2}\left(U_{n+1}(x)+U_{n-1}(x)\right).
\end{aligned} \tag{2.9}$$

Chebyshev polynomials zeros and extrema

The Chebyshev polynomials of degree $n>0$ of all four kinds have precisely n zeros and $n+1$ local extrema in the interval $[-1,1]$. Note that $n-1$ of these extrema are interior to $[-1,1]$ (in the sense that the gradient vanishes) and the other two extrema being at the end points ± 1(where the gradient is non-zero).

From formulas (2.1)-(2.2), the zeros for $x\in[-1,1]$ of $T_n(x)$ and $U_n(x)$ must correspond to the zeros for $\theta\in[0,\pi]$ of $\cos n\theta$ and $\sin(n+1)\theta$ so that the zeros of $T_n(x)$ and $U_n(x)$ are

$$\begin{aligned}
&x=x_k=\cos\frac{(2k-1)\pi}{2n}=\cos\frac{(n-k+1/2)\pi}{n},\ k=1,2,\ldots,n,\\
&x=x_k=\cos\frac{k\pi}{n+1}=\cos\frac{(n-k+1)\pi}{n+1},\ k=1,2,\ldots,n
\end{aligned} \tag{2.10}$$

The zeros of $V_n(x)$ and $W_n(x)$ occur at

$$\begin{aligned} x = x_k &= \cos\frac{(k-1/2)\pi}{n+1/2} = \cos\frac{(n-k+1/2)\pi}{n+1/2}, \ k = 1,2,\ldots,n, \\ x = x_k &= \cos\frac{k\pi}{n+1/2} = \cos\frac{(n-k+1)\pi}{n+1/2}, \ k = 1,2,\ldots,n. \end{aligned} \quad . \tag{2.11}$$

The second term in each equations of (2.10)-(2.11) is called their natural order.

The internal extrema of $T_n(x)$ correspond to the extrema of $\cos n\theta$, namely the zeros of $\sin n\theta$, since

$$\frac{d}{dx}T_n(x) = \frac{d}{dx}\cos n\theta = \frac{d}{d\theta}\cos n\theta \cdot \frac{d\theta}{dx} = \frac{d}{d\theta}\cos n\theta / \frac{dx}{d\theta} = \frac{n\sin n\theta}{\sin\theta} = U_{n-1}x\,. \tag{2.12}$$

Hence, including those at $x = \pm 1$, the extrema of $T_n(x)$ on $[-1,1]$ are

$$x = x_k = \cos\frac{k\pi}{n} = \cos\frac{(n-k)\pi}{n}, \ k = 0,1,2,\ldots,n, \,. \tag{2.13}$$

These are precisely the zeros of $(1-x^2)U_{n-1}(x)$. The extrema of $U_n(x), V_n(x), W_n(x)$ are not in general as readily determined indeed finding them involves the solution of transcendental equations. For example

$$\frac{d}{dx}U_n(x) = \frac{d}{dx}\frac{\sin(n+1)\theta}{\sin\theta} = \frac{-(n+1)\sin\theta\cos(n+1)\theta + \cos\theta\sin(n+1)\theta}{\sin^3\theta}\,. \tag{2.14}$$

On the other hand, from the Definitions 1.1-1.2 we can show that

$$\begin{aligned} &\sqrt{1-x^2}U_n(x) = \sin(n+1)\theta, \\ &\sqrt{1+x}V_n(x) = \sqrt{2}\cos(n+1/2)\theta, \,. \\ &\sqrt{1-x}W_n(x) = \sqrt{2}\sin(n+1/2)\theta. \end{aligned} \tag{2.15}$$

Hence the extrema of the weighted polynomials $\sqrt{1-x^2}U_n(x)$, $\sqrt{1+x}V_n(x)$, $\sqrt{1-x}W_n(x)$ are explicitely determined and occur, respectively at

$$\begin{aligned} x = x_k &= \cos\frac{(2k+1)\pi}{2(n+1)}, \ k = 0,1,2,\ldots,n, \\ x = x_k &= \cos\frac{2k\pi}{2n+1}, \ k = 0,1,2,\ldots,n, \\ x = x_k &= \cos\frac{(2k+1)\pi}{2n+1}, \ k = 0,1,2,\ldots,n. \end{aligned} \quad . \tag{2.16}$$

Evaluation of an integral

The indefinite integral of $T_n(x)$ can be expressed in terms of Chebyshev polynomials as follows

$$\begin{aligned}\int T_n(x)\,dx &= -\int \cos n\theta \sin\theta\, d\theta \\ &= -\frac{1}{2}\int\left[\sin(n+1)\theta - \sin(n+1)\theta\right]d\theta \\ &= \frac{1}{2}\left[\frac{\cos(n+1)\theta}{n+1} - \frac{\cos|n-1|\theta}{n-1}\right] \\ &= \begin{cases}\frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{|n-1|}(x)}{n-1}\right], & n \neq 1 \\ \frac{1}{4}T_2(x), & n = 1.\end{cases}\end{aligned} \quad . \tag{2.17}$$

Clearly this result can be used to integrate the sum

$$S_n(x) = \sum_{r=0}^{n}{}' a_r T_r(x).$$

In the form

$$\begin{aligned}I_{n+1}(x) &= \int S_n(x)\,dx = const + \frac{1}{2}a_0 T_1(x) + \frac{1}{4}a_1 T_2(x) \\ &+ \sum_{r=2}^{n}\frac{a_r}{2}\left[\frac{T_{r+1}(x)}{r+1} - \frac{T_{r-1}(x)}{r-1}\right] = \sum_{r=2}^{n} A_r T_r(x)\end{aligned} \quad , \tag{2.18}$$

where A_0 is determined from the const of integration and

$$A_r = \frac{a_{r-1} - a_{r+1}}{2r},\ r > 0,\ a_{n+1} = a_{n+2} = 0\,. \tag{2.19}$$

There is an interesting and direct integral relationship between the Chebyshev polynomials of the first and second kinds, namely

$$\int U_n(x)\,dx = \frac{1}{n+1}T_{n+1}(x) + C\,. \tag{2.20}$$

Hence the sum

$$\tilde{S}_n(x) = \sum_{r=1}^{n} b_r U_{r-1}(x).$$

Can be integrated immediately to give

$$\int \tilde{S}_n(x)\,dx = \sum_{r=1}^{n}\frac{b_r}{r}T_r(x) + C\,. \tag{2.21}$$

2.2. Orthogonality and Least-Square Approximation

From minimax to least squares

Orthogonal polynomials have a great variety and wealth of properties, many of which are noted in this section. Indeed, some of these properties take a very concise form in the case of the Chebyshev polynomials, making Chebyshev polynomials of leading importance among orthogonal polynomials – second perhaps to Legendre polynomials (which have a unit weight function), but having the advantage over the Legendre polynomials that the locations of their zeros are known analytically. The continuous and discrete orthogonality of the Chebeyshev polynomials may be viewed as a direct consequence of the orthogonality of sine and cosine functions of multiple angles, a central feature in the study of Fourier series.

Finally, the Chebyshev polynomials are orthogonal not only as polynomials in the real variable x on the real interval $[-1,1]$ but also as polynomials in a complex variable z on elliptical contours and domains of the complex plane (the foci of the ellipses being at -1 and $+1$). This property is exploited in fields such as crack problems in fracture mechanics (Gladwell & England 1977) and two-dimensional aerodynamics (Fromme & Golberg 1979, Fromme & Golberg 1981), which rely on complex-variable techniques. More generally, however, many real functions may be extended into analytic functions, and Chebyshev polynomials are remarkably robust in approximating on $[-1,1]$ functions which have complex poles close to that interval. This is a consequence of the fact that the interval $[-1,1]$ may be enclosed in an arbitrarily thin ellipse which excludes nearby singularities.

2.3 Orthogonality of Chebyshev polynomials (Orthogonal polynomials and weight functions)

Definition 1.3: *Two functions $f(x)$ and $g(x)$ in $\mathcal{L}_2[a,b]$ are said to be orthogonal on the interval $[a,b]$ with respect to a given continuous and non-negative weight function $w(x)$ if*

$$\int_a^b w(x)\, f(x) g(x) dx = 0\,, \tag{2.22}$$

if we use the inner product notations

$$\langle f, g \rangle = \int_a^b w(x)\, f(x) g(x) dx \tag{2.23}$$

where w, f and g are functions of x on $[a,b]$, then the orthogonality condition (2.22) is equivalent to saying that f is orthogonal to g if

$$\langle f, g \rangle = 0\,. \tag{2.24}$$

The formal definition of an inner product (in the context of real functions of a real variable – see Definition 4.3 for the complex case) is as follows:

Definition 4.2: *An inner product $\langle \cdot,\cdot \rangle$ is a bilinear function of elements $f, g, h, \ldots$ of a vector space that satisfies the axioms:*

1. $\langle f, f\rangle \geq 0$ *with equality if and only if* $f \equiv 0$;
2. $\langle f, g\rangle = \langle g, f\rangle$;
3. $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$;
4. $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ *for any scalar* α.

An inner product defines an $\mathcal{L}_2$ *– type norm*

$$\|f\| = \|f\|_2 := \sqrt{\langle f, f\rangle}. \tag{2.25}$$

Here we shall be concerned with families of orthogonal polynomials $\{\emptyset_i(x), i = 0,1,2, \ldots\}$ where $\emptyset_i$ is of degree i exactly, defined so that

$$\langle \emptyset_i, \emptyset_j\rangle = 0 \ \ (i \neq j). \tag{2.26}$$

Clearly, since $w(x)$ is non-negative,

$$\langle \emptyset_i, \emptyset_i\rangle = \|\emptyset_i\|^2 > 0. \tag{2.27}$$

The requirement that $\emptyset_i$ should be of exact degree i, together with the orthogonality condition (2.26), defines each polynomial $\emptyset_i$ uniquely apart from a multiplicative constant. The definition may be made unique by fixing the value of $\langle \emptyset_i, \emptyset_i\rangle$ or of its square root $\|\emptyset_i\|$. In particular, we say that the family is *orthonormal* if, in addition to (2.26), the functions $\{\emptyset_i(x)\}$ satisfy

$$\|\emptyset_i\| = 1 \ \ \text{for all} \ \ i. \tag{2.28}$$

2.4 Chebyshev polynomials as orthogonal polynomials

If we define the inner product (2.23) using the interval and weight function

$$[a, b] = [-1,1], \ \ w(x) = (1 - x^2)^{-\frac{1}{2}}, \tag{2.29}$$

then we find that the first kind Chebyshev polynomials satisfy

$$\langle T_i, T_j\rangle = \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos i\theta \cos j\theta d\theta, \tag{2.30}$$

(shown by setting $x = \cos\theta$ and using the relation $T_i(x) = \cos i\theta$ and $dx = -\sin\theta\, d\theta = -\sqrt{1 - x^2}\, d\theta$). Now, for $i \neq j$,

$$\begin{aligned}\int_0^{\pi} \cos(i\theta)\cos(j\theta) d\theta &= \frac{1}{2}\int_0^{\pi} [\cos(i+j)\theta] + [\cos(i-j)\theta] d\theta \\ &= \frac{1}{2}\left[\frac{\sin(i+j)\theta}{i+j} + \frac{\sin(i-j)\theta}{i-j}\right]_0^{\pi} = 0.\end{aligned}$$

Hence

$$\langle T_i, T_j\rangle = 0 \qquad (i \neq j), \tag{2.31}$$

and $\{T_i(x), i=0,1,...\}$ form an orthogonal polynomial system on $[-1,1]$ with respect to the weight $(1-x^2)^{-\frac{1}{2}}$.

The norm of T_i is given by

$$\begin{aligned}\|T_i\|^2 &= \langle T_i, T_i\rangle = \int_0^\pi (\cos i\theta)^2\, d\theta = \frac{1}{2}\int_0^\pi 1+\cos 2i\theta d\theta \\ &= \frac{1}{2}\left[\theta + \frac{\sin 2i\theta}{2i}\right]_0^\pi = \frac{1}{2}\pi,\end{aligned} \tag{2.32}$$

while

$$\|T_0\|^2 = \langle T_0, T_0\rangle = \langle 1,1\rangle = \pi. \tag{2.33}$$

The system $\{T_i\}$ is therefore not orthonormal. We could, if we wished, scale the polynomials to derive the orthonormal system

$$\sqrt{\frac{1}{\pi}}T_0(x), \left\{\sqrt{\frac{2}{\pi}}T_i(x), i=1,2,...\right\},$$

but the resulting irrational coefficients usually make this inconvenient. It is simpler in practice to adopt the $\{T_i\}$ we defined initially, taking note of the values of their norms (2.32)-(2.33).

The second, third and fourth kind Chebyshev polynomials are also orthogonal systems on $[-1,1]$, with respect to appropriate weight functions:

- $U_i(x)$ are orthogonal with respect to $w(x) = (1-x^2)^{\frac{1}{2}}$;
- $V_i(x)$ are orthogonal with respect to $w(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$;
- $W_i(x)$ are orthogonal with respect to $w(x) = (1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$.

These results are obtained from trigonometric relations as follows:

$$\begin{aligned}\langle U_i, U_j\rangle &= \int_{-1}^1 (1-x^2)^{\frac{1}{2}} U_i(x)U_j(x)dx \\ &= \int_{-1}^1 (1-x^2)^{-\frac{1}{2}}(1-x^2)^{\frac{1}{2}} U_i(x)(1-x^2)^{\frac{1}{2}} U_j(x)dx \\ &= \int_0^\pi \sin(i+1)\theta \sin(j+1)\theta d\theta \\ &\quad (\text{Since } \sin\theta\, U_i(x) = \sin(i+1)\theta) \\ &= \frac{1}{2}\int_0^\pi \cos(i-j)\theta - \cos(i+j+2)\theta d\theta = 0,\ (i \neq j).\end{aligned}$$

$$\begin{aligned}\langle V_i, V_j\rangle &= \int_{-1}^{1}(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}V_i(x)V_j(x)\,dx\\ &= \int_{-1}^{1}\left(1-x^2\right)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}V_i(x)(1+x)^{\frac{1}{2}}V_j(x)\,dx\\ &= 2\int_0^{\pi}\cos\left(i+\frac{1}{2}\right)\theta\cos\left(j+\frac{1}{2}\right)\theta\,d\theta\end{aligned}$$

Since $(1+x)^{\frac{1}{2}} = (1+\cos\theta)^{\frac{1}{2}} = \left(2\cos^2\frac{1}{2}\theta\right)^{\frac{1}{2}} = \sqrt{2}\cos\frac{1}{2}\theta$ and $(1+x)^{\frac{1}{2}}V_i(x) = \sqrt{2}\cos\left(i+\frac{1}{2}\right)\theta$ we have

$$\langle V_i, V_j\rangle == \int_0^{\pi}\cos(i+j+1)\theta+\cos(i-j)\theta\,d\theta = 0,\ (i\neq j).$$

Now

$$\begin{aligned}\langle W_i, W_j\rangle &= \int_{-1}^{1}(1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}W_i(x)W_j(x)\,dx\\ &= \int_{-1}^{1}\left(1-x^2\right)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}W_i(x)(1-x)^{\frac{1}{2}}W_j(x)\,dx\\ &= 2\int_0^{\pi}\sin\left(i+\frac{1}{2}\right)\theta\sin\left(j+\frac{1}{2}\right)\theta\,d\theta\end{aligned}$$

Since $(1-x)^{\frac{1}{2}} = (1-\cos\theta)^{\frac{1}{2}} = \left(2\sin^2\frac{1}{2}\theta\right)^{\frac{1}{2}} = \sqrt{2}\sin\frac{1}{2}\theta$ and $(1-x)^{\frac{1}{2}}W_i(x) = \sqrt{2}\sin\left(i+\frac{1}{2}\right)\theta$ we have

$$\langle W_i, W_j\rangle = \int_0^{\pi}\cos(i-j)\theta-\cos(i+j+1)\theta\,d\theta = 0,\ (i\neq j).$$

The normalizations that correspond to these polynomials are as follows (for all $i\geq 0$):

$$\langle U_i, U_i\rangle = \|U_i\|^2 = \int_0^{\pi}\sin^2(i+1)\theta\,d\theta = \frac{1}{2}\pi; \tag{2.34}$$

$$\langle V_i, V_i\rangle = \|V_i\|^2 = 2\int_0^{\pi}\cos^2\left(i+\frac{1}{2}\right)\theta\,d\theta = \pi; \tag{2.35}$$

$$\langle W_i, W_i\rangle = \|W_i\|^2 = 2\int_0^{\pi}\sin^2\left(i+\frac{1}{2}\right)\theta\,d\theta = \pi. \tag{2.36}$$

(Remember that each of these three identities uses a different definition of the inner product $\langle .,.\rangle$, since the weights $w(x)$ differ.)

2.5 Orthogonal polynomials and best L_2 approximations

Let us consider the best L_2 polynominal approximation of a given degree, which leads us to an orthogonality property.

The theorems in this section are valid not only for the inner product (4.2) but for any inner product $\langle .,. \rangle$ as defined by Definition 4.2

Theorem 4.3: The best L_2 polynomial $p_n^B(x)$ of degree (or less) to a given (L_2-integrable) function $f(x)$ is unique and is characterized by the (necessary and sufficient) property that

$$\langle f - p_n^B, p_n \rangle = 0 \tag{2.37}$$

for any other polynomial p_n of degree *n*.

Proof:

Part 1, (Necessity): Write $e_n^B := f - p_n^B$. Suppose that, for some polynomial p_n

$$\langle e_n^B, p_n \rangle \neq 0$$

Then, for any real scalar multiplier λ,

$$\begin{aligned}
\| f - (p_n^B + \lambda p_n \|^2 &= \| e_n^B - \lambda p_n \|^2 \\
&= \langle e_n^B - \lambda p_n, e_n^B - \lambda p_n \rangle \\
&= \langle e_n^B, e_n^B \rangle - 2\lambda \langle e_n^B, p_n \rangle + \lambda^2 \langle p_n, p_n \rangle \\
&= \| e_n^B \|^2 - 2\lambda \langle e_n^B, p_n \rangle + \lambda^2 \| p_n \|^2 < \| e_n^B \|^2
\end{aligned}$$

for some small λ of the same sign as $\langle e_n^B, p_n \rangle$. Hence $p_n^B + \lambda p_n$ is a better approximation than p_n^B for this value of λ,contradicting the assertion that p_n^B is a best approximation.

Part 2, (Sufficiency): Suppose that (2.37) holds and that q_n is any specified polynomial of degree ***n,*** not identical to p_n^B.Then

$$\begin{aligned}
\| f - q_n \|^2 - \| f - p_n^B \|^2 &= \| e_n^B + (p_n^B - q_n) \|^2 - \| e_n^B \|^2 \\
&= \langle e_n^B + (p_n^B - q_n), e_n^B + (p_n^B + q_n) \rangle - \langle e_n^B, e_n^B \rangle \\
&= \langle p_n^B - q_n, p_n^B - q_n \rangle + 2 \langle e_n^B, p_n^B - q_n \rangle \\
&= \| p_n^B - q_n \|^2 + 0
\end{aligned}$$

From (2.37), therefore $\| f - q_n \|^2 > \| f - q_n \|^2$.

Since q_n is arbitrary, p_n^B must be a best L_2 approximation. It must also be unique, since otherwise we could have taken q_n to be another best approximation and obtained the last inequality as a contradiction.

Corollary 4.1 If $\{\phi_n\}$ being exact degree *i* and an orthogonal polynomial system on [a, b] then:

1) The zero function is the best L_2 polynomial approximation of degree (n-1) to $\{\phi_n\}$ on [a,b].

2) $\{\phi_n\}$ is the best L_2 approximation to zero on [a,b] among polynomials of degree n with the same leading coefficient.

Proof:

1. Any polynomial p_{n-1} of degree n-1 can be written in the form

$$p_{n-1} = \sum_{i=0}^{n-1} c_i \phi_i.$$

Then

$$\langle \phi_n - 0, p_{n-1} \rangle = \left\langle \phi_n, \sum_{i=0}^{n-1} c_i, \phi_i \right\rangle = \sum_{i=0}^{n-1} c_i \langle \phi_i, \phi_n \rangle = 0$$

by the orthogonality of $\{\phi_i\}$. The result follows from Theorem 4.1.

2) Let q_n be any other polynomial of degree n having the same leading coefficient as $\{\phi_n\}$.Then -ϕ_n is a polynomial of degree n-1 .We can therefore write

$$q_n - \phi_n = \sum_{i=0}^{n-1} c_i, \phi_i$$

and deduce from the orthagonality of $\{\phi_i\}$ that

$$\langle \phi_n, q_n - \phi_n \rangle = 0$$

Now we have

$$\begin{aligned} \|q_n\|^2 - \|\phi_n\|^2 &= \langle q_n, q_n \rangle - \langle \phi_n, \phi_n \rangle \\ &= \langle q_n - \phi_n, q_n - \phi_n \rangle - 2\langle \phi_n, q_n - \phi_n \rangle \\ &= \|q_n - \phi_n\|^2 > 0, \text{ using (4.16)} \end{aligned}$$

Therefore ϕ_n is the best approximation to zero .

The interesting observation that follows from Corollary 4.1 A is that every polynomial in an orthogonal system has a minimal L_2 property - analogous to the minimax property of the Chebyshev polynomials. Indeed, the four kinds of Chebyshev polynomials T_n, U_n, V_n, W_n, being orthogonal polynomials each have a minimal property on [-1,1] with respect to their respective weight functions

$$\frac{1}{\sqrt{1-x^2}}, \sqrt{1-x^2}, \sqrt{\frac{1+x}{1-x}}, \sqrt{\frac{1-x}{1+x}},$$

over all polynomials with the same leading coefficients.

The main result above,namely Theorem 4.1 is essentially a generalization of the statement that the shortest distance from a point to a plane is in the direction of a vector perpendicular to all vectors in that plane.

Theorem 4.1 is important in that it leads to a very direct algorithm for determining the best L_2 polynomial approximation p_n^B to f :

Corollary 4.1 B The best L_2 polynomial approximation p_n^B of degree n to f

$$q_n^B = \sum_{i=0}^{n-1} c_i, \phi_i$$

where $c_i = \frac{\langle f, \phi_i \rangle}{\langle \phi_{i,} \phi_{i,} \rangle}$.

Proof: For $k = 0,1...,n$

$$\begin{aligned} \langle f - p_n^B, \phi_k \rangle &= \left\langle f - \sum_{i=0}^{n} c_i \phi_i, \phi_k \right\rangle \\ &= \langle f - \phi_k \rangle - \sum_{i=0}^{n} c_1 \langle \phi_i, \phi_k \rangle \qquad (2.38) \\ &= \langle f - \phi_k \rangle - c_k \langle \phi_i, \phi_k \rangle = 0, \text{ by definition of } c_k \end{aligned}$$

Now, any polynomial p_n can be written as

$$p_n = \sum_{i=0}^{n} d_i \phi_i$$

and hence

$$\langle f - p_n^B, p_n \rangle = \sum_{i=0}^{n} d_i \langle f - p_n^B, \phi_i \rangle = 0 \text{ by } (17)$$

Thus p_n^B is the best approximation by Theorem 4.1

Example 4.1: To illustrate Corollary 4.1 B, suppose that we wish to determine the best L_2 linear approximate p_1^B to $f(x) = 1 - x^2$ on [-1,1] with respect to the weight $w(x) = (1-x^2)^{-\frac{1}{2}}$ .In this case $\{T_i(x)\}$ is the appropriate orthogonal system and hence

$$p_1^B = c_0 T_0(x) + c_1 T_1(x)$$

where by (4.17)

$$c_0 = \frac{\langle f, T_0 \rangle}{\langle T_0, T_0 \rangle} = \frac{\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}}(1-x^2)dx}{\pi}$$

$$c_1 = \frac{\langle f, T_0 \rangle}{\langle T_0, T_0 \rangle} = \frac{\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}}(1-x^2)dx}{\frac{1}{2}\pi}$$

Substituting $x = \cos\theta$,

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin^2\theta d\theta = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2\theta)d\theta = \frac{1}{2},$$

$$c_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2\theta\cos\theta d\theta = \frac{2}{\pi}[\tfrac{1}{3}\sin^3\theta]_0^{\pi}$$

and therefore

$$p_1^B = \tfrac{1}{2}T_0(x) + 0T_1(x) = \tfrac{1}{2},$$

so that the linear approximation reduces to a constant in this case.

2.6 Orthogonal polynomial expansions

On the assumption that it is possible to expand a given function $f(x)$ in a (suitably convergent) series based on a system $\{\phi_k\}$ of polynomials orthogonal over the interval [a,b] $\{\phi_k\}$ being of exact degree k ,we may write

$$f(x) = \sum_{k=0}^{\infty} c_i, \phi_i(x),\ x \in [a,b]. \tag{2.39}$$

It follows, by taking inner products with ϕ_k , that

$$\langle f, \phi_k \rangle = \sum_{k=0}^{\infty} c_1 \langle \phi_{1,} \phi_k \rangle = c_k \langle \phi_k, \phi_k \rangle,$$

Since $\langle \phi_i, \phi_k \rangle = 0$ for $i \neq k$. This is identical to the formula for c_k given in Corollary 4.1B.Thus (applying the same corollary)an orthogonal expansion has the property that its partial sum of degree n is the best L_2 approximation of degree n to its infinite sum.Hence it is an ideal expansion to use in the L_2 context.In particular,the four Chebyshev series expansions have this property on [-1,1] with respect to their respective weight functions $(1+x)^{\pm\frac{1}{2}}(1-x)^{\pm\frac{1}{2}}$.

2.7 Convergence in L_2 of orthogonal expansions

Convergence questions will be considered in detail in Chapter 5, where we shall restrict attention to Chebyshev polynomials and use Fourier series theory.However, we may easily make some deductions from general orthogonal polynomial properties.

In particular, if f is continuous, then we know (Theorem 3.2) that arbitrarily accurate polynomial approximations exist in $C[a,b]$, and it follows from Lemma 3.1 that these are also arbitrarily accurate in L_2 **[a,b].** However, we have shown in Section 4.3.1 that the *n*th degree polynomial, $P_n(x)$ say, obtained by truncating an orthogonal polynomial expansion is a best L_2 approximation. Hence (a fortiori) P_n must also achieve an arbitrarily small L_2 error $\|f - P_n\|_2$ for sufficiently large n.This gives the following result.

Theorem 1.3 If f is in $C[a,b]$, ,then its expansion in orthogonal polynomials converges in L_2 (with respect to the appropriate weight function).

In Chapter 5, we obtain much more powerful convergence results for Chebyshev series, ensuring L_2 convergence of the series itself for f in L_2 **[a,b]** and L_∞ convergence of Cesaro sums of the series for f in $C[a,b]$,

Recurrence relations

Using the inner product (4.2) namely

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx$$

we note that

$$\langle f, g \rangle = \langle g, f \rangle$$
$$\langle xf, g \rangle = \langle f, xg \rangle$$

The following formulae uniquely define an orthogonal polynomial system $\{\phi_i\}$, in which ϕ_i is a monic polynomial (i.e., a polynomial with a leading coefficient of unity) of exact degree i.

Theorem 1.3 The unique system of monic polynomials $\{\phi_i\}$ with ϕ_i of exact degree i, which are orthogonal on [a,b] with respect to $w(x)$ are defined by

$$\begin{aligned} &\phi_0(x) = 1, \\ &\phi_1(x) = x - a_1, \\ &\phi_n(x) = (x - a_n)\phi_n - 1(x) - b_n\phi_n - 2(x), \end{aligned} \tag{2.40}$$

where

$$a_n = \frac{\langle x\phi_{n-1,}\phi_{n-1,}\rangle}{\langle \phi_{n-1,}, \phi_{n-1,}\rangle}, \qquad b_n = \frac{\langle \phi_{n-1}, \phi_{n-1}\rangle}{\langle \phi_{n-2}, \phi_{n-2}\rangle}$$

Proof: This is readily shown by induction on n .It is easy to shown that the polynomials ϕ_n generated by (4.21) are all monic.We assume that the polynomials $\phi_0, \phi_1, ...\phi_{n-1}$ are orthogonal,and we then need to test that ϕ_n ,as given by (4.21) is orthogonal to $\phi_k (k = 0,1,....n-1)$.

The polynomial $x\phi_k$ is a monic polynomial of degree $k+1$, expressible in the form

$$x\phi_k(x) = \phi_{k+1(x)} + \sum_{i=1}^{k} c_i \phi_i(x),$$

so that using (4.20),

$$\langle x\phi_{n-1}, \phi_k\rangle = \langle \phi_{n-1}, x\phi_k\rangle = 0 (k < n-2),$$
$$\langle x\phi_{n-1}, \phi_{n-2}\rangle = \langle \phi_{n-1}, x\phi_{n-2}\rangle = \langle \phi_{n-1}, \phi_{n-1}\rangle.$$

For k<n-2, then we have

$$\langle \phi_n, \phi_k\rangle = \langle x\phi_{n-1}, \phi_k\rangle - a_n \langle \phi_n, \phi_k\rangle - b_n \langle \phi_{n-2}, \phi_k\rangle = 0$$

While

$$\langle \phi_n, \phi_{n-2}\rangle = \langle x\phi_{n-1}, \phi_{n-2}\rangle - a_n \langle \phi_{n-1}, \phi_{n-2}\rangle - b_n \langle \phi_{n-2}, \phi_{n-2}\rangle$$
$$= \langle \phi_{n-1}, \phi_{n-1}\rangle - 0 - \langle \phi_{n-1}, \phi_{n-1}\rangle = 0,$$
$$\langle \phi_n, \phi_{n-1}\rangle = \langle x\phi_{n-1}, \phi_{n-1}\rangle - a_n \langle \phi_{n-1}, \phi_{n-1}\rangle - b_n \langle \phi_{n-2}, \phi_{n-1}\rangle$$
$$= \langle x\phi_{n-1}, \phi_{n-1}\rangle - \langle x\phi_{n-1}, \phi_{n-1}\rangle - 0 = 0.$$

Starting the induction is easy and the result follows.

We have already established a recurrence relation for each of the four kinds of Chebyshev polynomials.We can verify that (4.21) leads to the same recurrences.

Consider the case of the polynomials of the first kind.We convert $T_n(x)$ to a monic polynomial by writing $\phi_0 = T_0, \phi_n = 2^{1-n} T_n (n > 0)$. Then we can find the inner products:

$$\langle T_0, T_0 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\pi} d\theta = \pi,$$

$$\langle xT_0, T_0 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos\theta d\theta = 0$$

$$\langle T_n, T_n \rangle = \int_{-1}^{1} \frac{T_n(x)^2}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos^2\theta d\theta = \frac{1}{2}\pi,$$

$$\langle xT_n, T_n \rangle = \int_{-1}^{1} \frac{xT_n(x)^2}{\sqrt{1-x^2}} dx = \int_0^{\pi} \cos\theta \cos^2 n\theta d\theta = 0$$

Therefore $a_1 = 0, a_n = 0 (n > 1),$ and

$$b_2 = \frac{\langle \phi_1, \phi_1 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\langle T_1, T_1 \rangle}{\langle T_0, T_0 \rangle} = \frac{1}{2},$$

$$b_2 = \frac{\langle \phi_{n-1}, \phi_{n-1} \rangle}{\langle \phi_{n-2}, \phi_{n-2} \rangle} = \frac{\langle 2^{2-n} T_{n-1}, 2^{2-n} T_{n-1} \rangle}{\langle 2^{3-n} T_{n-2}, 2^{3-n} T_{n-2} \rangle} = \frac{1}{4} (n > 2).$$

So

$$\phi_0 = 1,$$
$$\phi_1 = x,$$
$$\phi_2 = x\phi_1 - \frac{1}{2}\phi_0$$
$$\phi_{n=} x\phi_{n-1} - \frac{1}{4}\phi_{n-2} (n > 2).$$

Hence the recurrence (1.3) for T_n. We may similarly derive the recurrences (1.6) for U_n and (1.2) for V_n and w_n, by using their respective weight functions to obtain the appropriate a_n and b_n (see Problem 5)

6.3 Chebyshev interpolation formulae

We showed in Section 4.6 that the Chebyshev polynomials $\{T_i(x)\}$ of degrees up to n are orthogonal in a discrete sense on the set (6.3) of zeros $\{x_k\}$ of $T_{n+1}(x)$.Specify

$$\sum_{k=1}^{n+1} T_i(x_k)\ T_j(x_k) = \begin{cases} 0 & i \neq j\ (\leq n) \\ n+1 & i = j = 0 \\ \frac{1}{2} & 0 < i = j \leq n \end{cases} \tag{2.41}$$

This discrete orthogonality property leads us to a very efficient interpolation formula.Write the nth degree polynomial $p_n(x)$,interpolating $f(x)$ in the points ,as a sum of Chebyshev polynomials in the form

$$p_n(x) = \sum_{i=0}^{n}{}' c_i T_i(x) \qquad (2.42)$$

Theorem 6.7: The coefficients c_i in (6.13) are given by the explicit formula

$$c_i = \frac{2}{n+1} \sum_{k=1}^{n+1} f(x_k) T_i(x_k). \qquad (2.43)$$

Proof: If we set equal to $p_n(x)$ at the points $\{x_k\}$,then it follows that

$$f(x_k) = \sum_{i=0}^{n}{}' f(x_k) T_i(x_k).$$

Hence,multiplying by $\frac{2}{n+1} T_i(x_k)$. and summing,

$$\frac{2}{n+1} \sum_{k=1}^{n+1} f(x_k) T_i(x_k) = \sum_{i=0}^{n}{}' c_i \left\{ \frac{2}{n+1} \sum_{k=1}^{n+1} T_i(x_k) T_j(x_k) \right\} = c_j$$

from (6.12) giving the formula (6.14).

Corollary 6.7 A Formula (6.14) is equivalent to a discrete Fourier transform of the transformed function

$$g(\theta) = f(\cos\theta).$$

Proof: We have $p_n(\cos\theta) = \sum_{i=0}^{n}{}' c_i \cos i\theta$ with

$$c_i = \frac{2}{n+1} \sum_{k=1}^{n+1} g(\theta_k) \cos i\theta_k \qquad (2.44)$$

with

$$\theta_k = \frac{(k - \frac{1}{2})\pi}{n+1}. \qquad (2.45)$$

Thus $\{c_i\}$ are discrete approximations to the true Fourier cosine series coefficients

$$c_i^S = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos i\theta \, d\theta, \qquad (2.46)$$

obtained by applying a trapezoidal quadrature rule to the (periodic) function $g(\theta)$ with equal intervals $\pi/(n+1)$ *between the points* θ_k .Indeed,a trapezoidal rule approximations to (6.17) ,valid for any periodic function $g(\theta)$,is

$$c_i^S \simeq \frac{1}{\pi}\frac{\pi}{n+1}\sum_{k=-n}^{n+1} g\left(\frac{(k-\frac{1}{2})\pi}{n+1}\right)\cos\frac{i(k-\frac{1}{2})\pi}{n+1},$$

which gives exactly the formula (6.15) for c_i (when we note that the fact that both $g(\theta)$ and $\cos i\theta$ are even functions implies that the k th and (1-k)th terms in the summation aer identical)

Second-kind interpolation

Consider in this case interpolation by a weighted polynomial $\sqrt{1-x^2}\ p_n(x)$ *on the zeros of* $U_{n+1}(x)$, namely

$$Y_k = \cos\frac{k\pi}{n+2} \quad (k=1,.....,n+1).$$

Theorem 6.8 the weighted interpolation polynomial to $f(x)$ is given by

$$\sqrt{1-x^2}\ p_n(x) = \sqrt{1-x^2}\sum_{i=0}^{n} c_i U_i(x), \qquad (2.47)$$

with coefficients given by

$$c_i = \frac{2}{n+1}\sum_{k=1}^{n+1}\sqrt{1-y_k^2}\, f(y_k)U_i(y_k). \qquad (2.48)$$

Proof: From (4.50) with n-1 replaced by n+1,

$$\sum_{k=1}^{n+1}(1-y_k^2)U_i(y_k)U_j(y_k) = \begin{cases} 0, & i \neq j(\leq n); \\ \frac{1}{2}(n+1), & i = j \leq n. \end{cases} \qquad (2.49)$$

If we set $\sqrt{1-y_k^2}\, p_n(y_k)$ *equal to* $f(y_k)$, *we obtain*

$$f(y_k) = \sqrt{1-y_k^2}\sum_{i=0}^{n} c_i U_i(y_k),$$

and hence, multiplying by $\frac{2}{n+1}\sqrt{1-y_k^2}\, U_j(y_k)$ and summing over k ,

$$\frac{2}{n+1}\sum_{k=1}^{n+1}\sqrt{1-y_k^2}\, f(y_k)U_j(y_k) = \sum_{i=0}^{n} c_1\left\{\frac{2}{n+1}\sum_{k=1}^{n+1}\left(1-y_k^2\right)U_i(y_k)U_j(y_k)\right\} = c_i$$

by (2.50)

Alternatively, we may want to interpolate at the zeros of $U_{n-1}(x)$ together with the points $x = \pm 1$, namely

$$y_k = \cos\frac{k\pi}{n} \quad (k = 0, \ldots, n).$$

In this case, however, we must express the interpolating polynomial as a sum of first-kind polynomials, when we can use the discrete orthogonality formula (4.45)

$$\sum_{k=0}^{n}{}'' T_i(y_k)T_j(x_k) = \begin{cases} 0, & i \neq j\ (\leq n); \\ \frac{1}{2}n, & 0 < i = j < n; \\ n, & i = j = 0; i = j = n. \end{cases} \qquad (2.51)$$

(Note the double prime indicating that the first and last terms of the sum are to be halved.)

The interpolating polynomial is then

$$p_n(x) = \sum_{i=0}^{n}{}'' c_i T_i(x) \qquad (2.52)$$

with coefficients given by

$$c_i = \frac{2}{n}\sum_{k=0}^{n}{}'' f(y_k)T_i(y_k). \qquad (2.53)$$

Apart from a factor of $\sqrt{2/n}$, these coefficients make up the discrete Chepyshev transform

Third and fourth-kind interpolation

Taking as interpolatin points the zeros of $V_{n=1}(x)$, namely

$$x_k = \cos\frac{\left(k - \frac{1}{2}\right)\pi}{n + \frac{3}{2}} \quad (k = 1, \ldots, n+1).$$

we have formula,for $i, j \leq n$,

$$\sum_{k=1}^{n+1}\left(1 + x_k\right)V_i(x_k)V_j(x_k) = \begin{cases} 0 & i \neq j \\ n + \frac{3}{2} & i = j \end{cases} \qquad (2.54)$$

Theorem 6.9 The weighted interpolation polynomial to $\sqrt{1+x}\,f(x)$ is given by

$$\sqrt{1+x}\,p_n(x) = \sqrt{1+x}\sum_{i=0}^{n} c_i V_i(x) \qquad (2.55)$$

where

$$c_i = \frac{1}{n + \frac{3}{2}}\sum_{k=1}^{n+1}\sqrt{1+x_k}\,f(x_k)V_i(x_k). \qquad (2.56)$$

Proof: If we set $\sqrt{1+x_k}\,p_n(x_k)$ equal to $\sqrt{1+x_k}\,f(x_k)$,we obtain

$$\sqrt{1+x_k}\, f(x_k) = \sqrt{1+x_k} \sum_{i=0}^{n} c_i V_i(x_k),$$

and hence,multiplying by $\frac{1}{n+\frac{3}{2}} \sqrt{1+x_k} V_i(x_k)$ and summing over k ,

$$\frac{1}{n+\frac{3}{2}} \sum_{k=1}^{n+1} (1+x_k) f(x_k) V_i(x_k) = \sum_{i=0}^{n} c_i \left\{ \frac{1}{n+\frac{3}{2}} \sum_{k=1}^{n+1} (1+x_k) f(x_k) V_i(x_k) \right\} = c_1 \qquad (2.57)$$

The same goes for interpolation at the zeros of $V_{n+1}(x)$,namely

$$x_k = \cos \frac{(n-k+2)\pi}{n+\frac{3}{2}} \quad (k = 1, \ldots.., n+1).$$

If we replace ‘V’ by ‘’W’’ and ‘ 1+x’ by ‘1-x’ throughout.

Alternatively,we may interpolate at the zeros of $V(x)$ together with one end point x=-1; i.e.., at the points

$$x_k = \cos \frac{(k-\frac{1}{2})\pi}{n+\frac{3}{2}}, \quad (k = 1, \ldots.., n+1).$$

where we have the discrete orthogonality formulae (the notation $\sum^{*}$ indicating that the last term of the summation is to halved)

$$\sum_{k=1}^{n+1}{}^{*} T_i(x_k)\ T_j(x_k) = \begin{cases} 0 & i \neq j\ (\leq n) \\ n+\frac{1}{2} & i = j = 0 \\ \frac{1}{2}(n+\frac{1}{2}) & 0 < i = j \leq n. \end{cases} \qquad (2.58)$$

The interpolating polynomial is then

$$p_n(x) = \sum_{i=0}^{n}{}' c_i T_i(x) \qquad (2.59)$$

with coefficients given by

$$c_i = \frac{2}{n+\frac{1}{2}} \sum_{k=1}^{n+1}{}^{*} f(x_k) T_i(x_k). \qquad (2.60)$$

Gauss-Chebyshev Quadrature Formula And Its Estimation

Gauss-Chebyshev quadrature formula (Mason [1, pp. 181-183])

Theorem 1: If $x_k, k = 1, \dots, n$ are the n zeros of $\emptyset_n(x)$, and $\{\emptyset_k(x), k = 0,1,2,..\}$ is the system of polynomials, $\emptyset_k(x)$ having the exact degree k, orthogonal with respect to $w(x)$ on $[a, b]$, then

$$\int_a^b w(x)f(x)dx \cong \sum_{k=1}^{n} A_k f(x_k), \tag{1}$$

$$A_k = \int_a^b w(x)l_k(x)dx, \quad l_k(x) = \prod_{\substack{r=1 \\ r\neq k}}^{n} \left(\frac{x-x_r}{x_k-x_r}\right), \tag{2}$$

quadrature formula (1) with coefficients (2) gives an exact result whenever $f(x)$ is a polynomial of degree 2n-1 or less. Moreover, all the coefficients A_k are positive in this case.

Theorem 2: (Mason [1, pp. 148]) If $\{x_k, k = 1,2,..,n+1\}$ are the zeros of polynomial $\{\emptyset_{n+1}(x)\}$, then the Lagrange polynomials defined by (2) may be written in the form

$$l_k(x) = \prod_{\substack{r=1 \\ r\neq k}}^{n} \left(\frac{x-x_r}{x_k-x_r}\right) = \frac{\emptyset_{n+1}(x)}{(x-x_k)\emptyset'_{n+1}(x)},$$

where $\emptyset'(x)$ denotes the derivative of $\emptyset(x)$.

Particularly, in the case of Chebyshev polynomials we have

Theorem 2: In the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} w(x)f(x)dx \cong \sum_{k=1}^{n} A_k f(x_k),$$

where $\{x_k\}$ are the n zeros of $\emptyset_n(x)$, the coefficients A_k are as follows:

1. For $w(x) = (1-x^2)^{-1/2}$, $\emptyset_n(x) = T_n(x)$, then $A_k = \frac{\pi}{n}$.
2. For $w(x) = (1-x^2)^{1/2}$, $\emptyset_n(x) = U_n(x)$, then $A_k = \frac{\pi}{n+1}(1-x_k^2)$.
3. For $w(x) = (1-x)^{-1/2}(1+x)^{1/2}$, $\emptyset_n(x) = V_n(x)$, then $A_k = \frac{\pi}{n+1/2}(1+x_k)$.
4. For $w(x) = (1-x)^{1/2}(1+x)^{-1/2}$, $\emptyset_n(x) = W(x)$, then $A_k = \frac{\pi}{n+1/2}(1-x_k)$.

Error estimation of truncated Chebyshev polynomials (Mason [1, pp. 131])

Let truncated Chebyshev polynomials be given by

$$S_n^T f(x) = \sum_{k=0}^{\prime n} c_k T_k(x), \quad c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}dx.$$

Theorem 3: If the function $f(x)$ has $r+1$ continuous derivative on [-1,1] then

$$|f(x) - S_n^T f(x)| = O(n^r),$$

for all $x \in [-1,1]$.

Error estimation of Gauss-Chebyshev quadrature formula (Kythe and Schaferkotter [2, pp. 109])

Theorem 4: If a function $f(x)$ has bounded derivative of order $2n$, then error term of Gaussian quadrature rule is given by

$$E_n(f) = \frac{\gamma_n}{a_n^2(2n)!} f^{(2n)}(\mu), \quad a < \mu < b,$$

where a_n is the leading coefficients of x^n and $\emptyset_n(x)$ is the orthogonal polynomials with respect to $w(x)$ on $[a, b]$ and

$$\gamma_n = \int_a^b w(x)\emptyset_n^2(x)dx.$$

Particularly, for the Gauss-Chebyshev QF of second kind we have

$$E_n(f) = \int_{-1}^{1} w(x)f(x)dx - \frac{\pi}{n+1}\sum_{k=1}^{n}\left(1 - x_k^2\right)f(x_k)$$

$$= \frac{\pi}{2^{2n+1}(2n)!} f^{(2n)}(\mu), \qquad -1 < \mu < 1.$$